Stabilization of DLA in a wedge

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Peking University

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Sichuan University

Joint work with:

Eviatar B. Procaccia

and

Ron Rosenthal

Special thanks to:

Wedge Antilles

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- **D**iffusion **L**imit **A**ggregation (DLA): first introduced by Witten and Sander in 1983.
- Simple model to study the geometry and dynamics of physical systems governed by diffusive laws:
- **Sparse** Bacterial growth.
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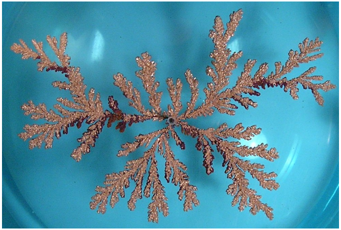

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- Discrete time, set-valued stochastic process $\{A_n\}_{n=0}^{\infty} \in \mathbb{Z}^2$, with $|A_n| = n$.
- $A_0 = \{0\}$. $A_{n+1} = A_n \cup \{y\}$, where $y \in \partial^{out} A_n$ is sampled according to $\mu_{\partial^{out} A_n}(\cdot)$, the **harmonic measure** on $\partial^{out} A_n$.
- For any $B \subset \mathbb{Z}^2$, let τ_B be the first time a simple random walk visiting B.
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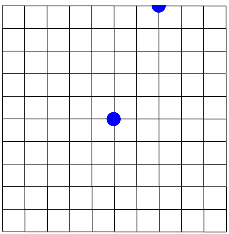

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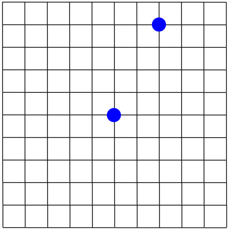

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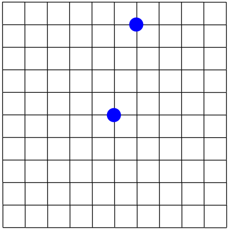

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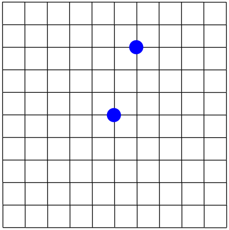

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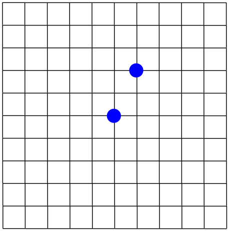

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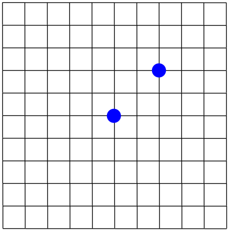

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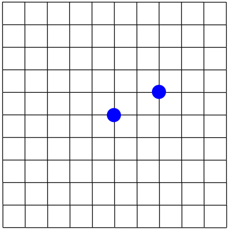

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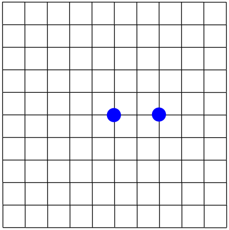

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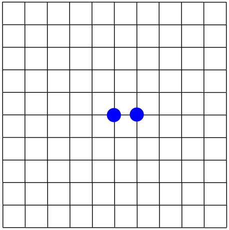

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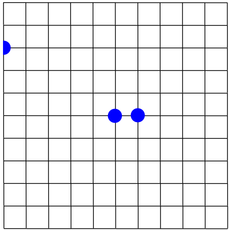

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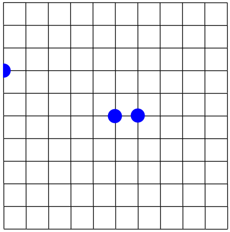

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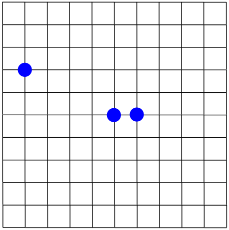

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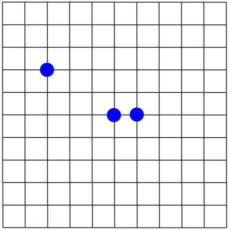

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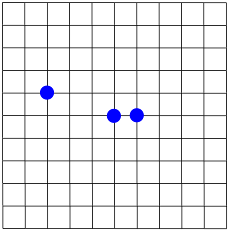

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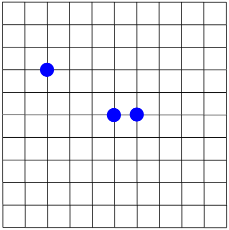

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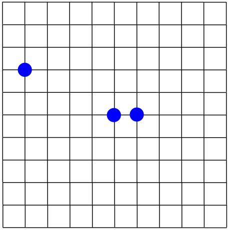

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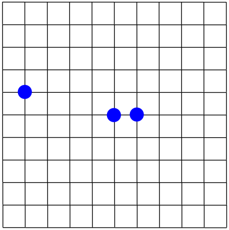

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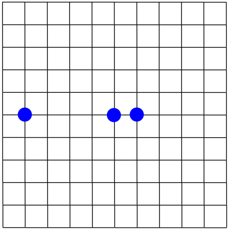

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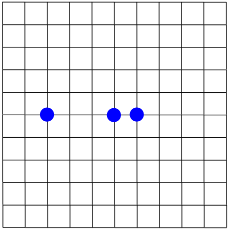

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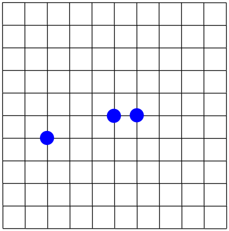

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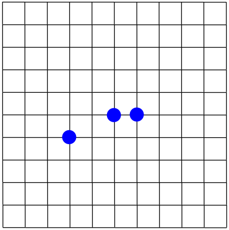

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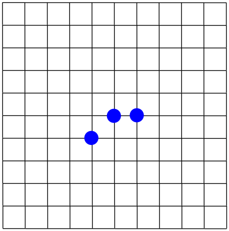

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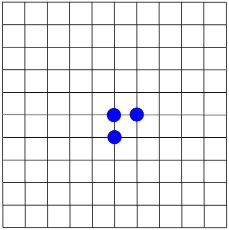

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- When $d = 2$, $\limsup_{n\to\infty} n^{-2/3}\|A_n\| \le C$.
- When $d \ge 3$, $\limsup_{n\to\infty} n^{-2/d}\|A_n\| \le C_d$.

In 1990, he improved the upper bounds to

- When $d = 3$, $\limsup_{n\to\infty}[n\log(n)]^{-1/2}\|A_n\| \le C_3$
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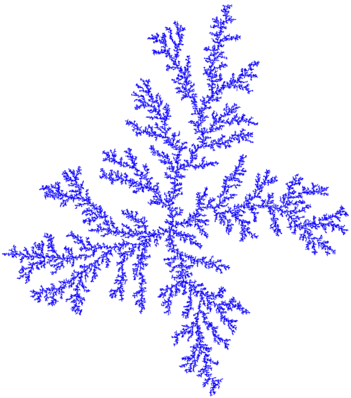

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- Do we know what it locally looks like? **No**
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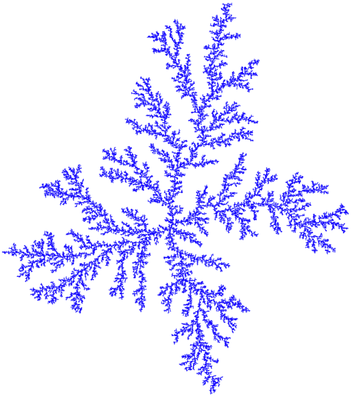

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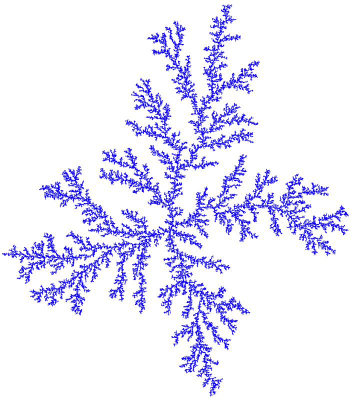

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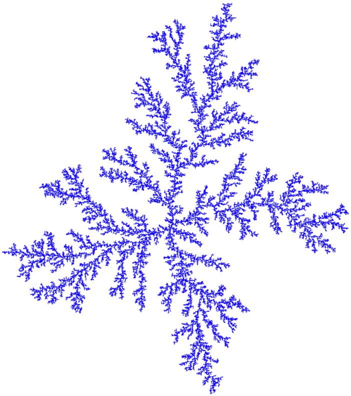

- Is there a infinite object? Yes
- Do we know what it locally looks like? **No**
- Wedge: *"It is impossible, even with a computer"*

Recent Work on DLA Upper Bounds

- In 2017, Benjamini and Yadin revisited this topic. They "clean up Kesten's argument", and generalize the original result to different types of graphs.

graph condition	upper bound	reference
transitive of polynomial growth, $d \geq 4$	$t^{2/d}$	Theorem 5.2
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Harmonic Measure in a wedge

In 2018, we study DLA in a wedge

$$W_{\theta_1,\theta_2} = \left\{(x,y) \in \mathbb{Z}^2 : \arctan(y/x) \in [\theta_1, \theta_2]\right\} ,$$

where $-\pi/2 \leq \theta_1 < \theta_2 \leq \pi/2$. I.e., new particles are added according to the harmonic measure given by the hitting probability of a random walk **in the wedge** starting from infinity.

Theorem 1 (Procaccia, Rosenthal and Z, 2018)

For a point $x \in W_{\theta_1,\theta_2}$, a set $A \subset W_{\theta_1,\theta_2}$, and $y \in A$ define $\mathcal{H}_A(x,y) = \mathbf{P}^x_{\theta_1,\theta_2}(S_{\tau_A} = y)$. For every $A \subset W_{\theta_1,\theta_2}$ and $y \in \mathbb{Z}^2$, the following limit, called the harmonic measure of A from infinity exists

$$\mathcal{H}_A^\infty(y) := \lim_{|x|\to\infty} \mathcal{H}_A(x,y).$$

Harmonic Measure in a wedge

Idea of the proof:

- Starting from radius R, it will take much more than R^2 steps for a random walk (in a wedge) to first visit A, which has a finite radius.
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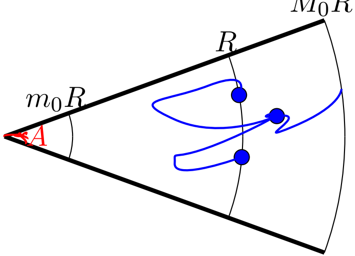

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A Beurling estimate theorem

For $R > 0$ let $B_R = \{(x, y) \in \mathbb{Z}^2 \ : \ x^2 + y^2 < R^2\}$ be the discrete Euclidean ball of radius R around the origin and define $W^R_{\theta_1,\theta_2} = W_{\theta_1,\theta_2} \cap B_R$.

Theorem 2 (Procaccia, Rosenthal and Z, 2018)

Fix $-\pi/2 \leq \theta_1 < \theta_2 \leq \pi/2$. For every $\epsilon > 0$, there exists $M \in \mathbb{N}$ and $C \in (0, \infty)$ such that for every $r, L \in \mathbb{N}$ satisfying $r \geq M$ and $L/r \geq M$, every $R > 0$ sufficiently large (depending on ϵ and L), every connected subset $A \subset W_{\theta_1,\theta_2}$, such that $W^r_{\theta_1,\theta_2} \subset A$ and that $A \cap \partial W^L_{\theta_1,\theta_2} \neq \emptyset$ and every $x \in \partial W^R_{\theta_1,\theta_2}$

$$\mathbf{P}^x_{\theta_1,\theta_2}\left(\tau_{\partial W^r_{\theta_1,\theta_2}} \leq \tau_{\partial A}\right) \leq C \left(\frac{r}{L}\right)^{\frac{\pi}{2(\theta_2-\theta_1)}-\epsilon} r \log L \qquad (1)$$

A Beurling estimate theorem

- Use time reversibility to replace the hitting probability with an escaping probability.
- For any $x \in \partial W^r_{\theta_1,\theta_2}$, replacing A by $W^r_{\theta_1,\theta_2}$ plus one of the two "walls" of the wedge increases the escaping probability.

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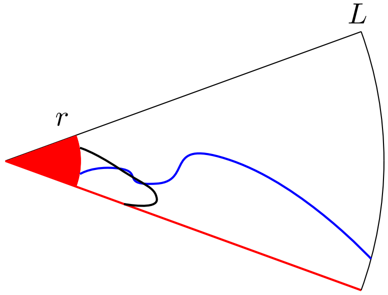

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Idea of the proof (in a continuous context):

- To find the correct conformal mapping to open the wedge, we have:
  - Option 1: $\varphi(z) = z^{2\pi/(\theta_2-\theta_1)}$

Dirichlet and Neumann boundaries meet!

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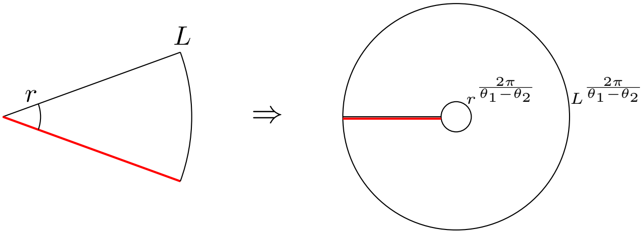

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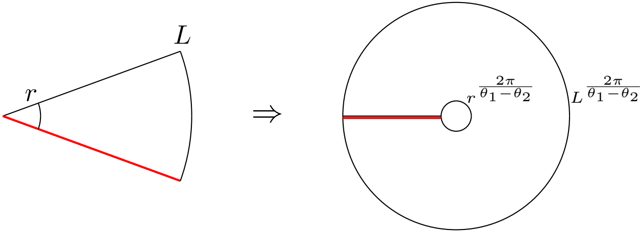

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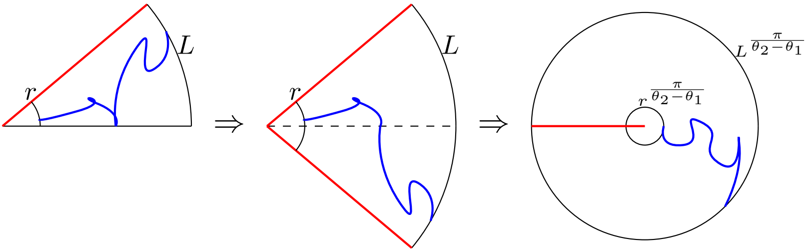
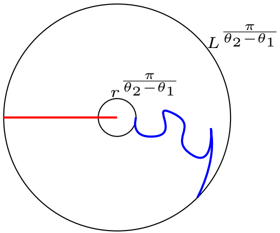
- Option 2: Reflection plus $\varphi(z) = z^{\pi/(\theta_2-\theta_1)}$



A Beurling estimate theorem

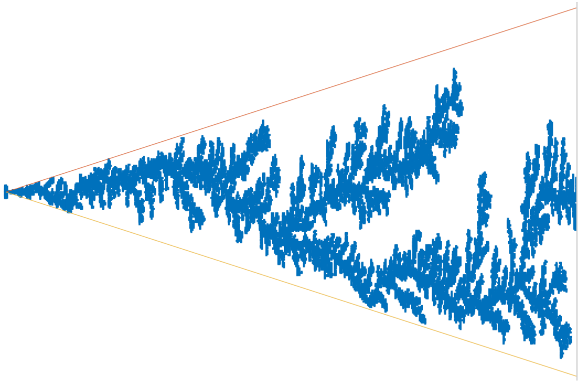
By (continuous) Beurling estimate, the probability of



is of order $\left(\frac{r}{L}\right)^{\frac{\pi}{2(\theta_2-\theta_1)}}$.

Intuition: It is more difficulty for random walk to escape a sharper wedge.

Stabilization of DLA in a wedge



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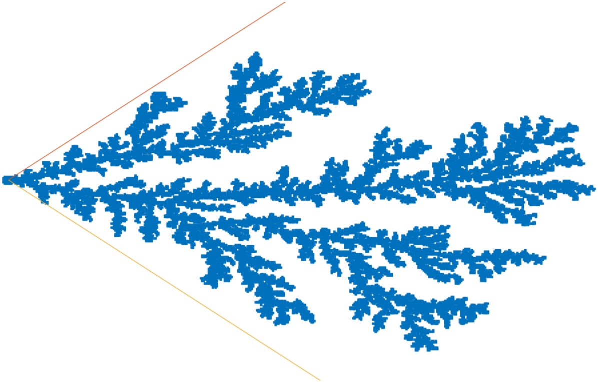

Stabilization of DLA in a wedge

Our main result is the stabilization of the DLA in sufficiently sharp wedges.

Theorem 3 (Procaccia, Rosenthal and Z, 2018)

Assume $-\pi/2 \leq \theta_1 < \theta_2 \leq \pi/2$ *satisfy* $\theta_2 - \theta_1 < \pi/4$ *and fix* $a > \frac{2\pi + 4(\theta_2 - \theta_1)}{\pi - 4(\theta_2 - \theta_1)}$. *Then* $\mathbb{P}_{\theta_1,\theta_2}$*-almost surely, for every* $R > 0$ *sufficiently large, the random sets* $(A_n \cap B_R)_{n \geq R^a}$ *are all the same. In other words, for all* R *sufficiently large, none of the particles* $(a_n)_{n \geq R^a}$ *added to the system after time* R^a *will attach to the aggregate inside* $W^R_{\theta_1,\theta_2}$.

This tells us, locally a long time simulation in a wedge will, with high probability, give the same configuration as in the infinite time DLA!

Open Problems

Can we prove stabilization of the DLA when $\theta_2 - \theta_1 = \pi/4$?

Conjecture 1

DLA in a wedge stabilizes when $\theta_2 - \theta_1 = \pi/4$.

If we can have the conjecture above, is it possible to use reflection symmetry of this special angle to have:

Conjecture 2

DLA in $\mathbb{Z}^2$ stabilizes.

Open Problems

Let $\gimel$ be an infinite graph. The number of ends of $\gimel$ is defined to be the supremum on the number of infinite, connected components of $\gimel \setminus K$, where we run over all finite $K \subset \gimel$. Hence, one can define the number of arms of the DLA as the number of ends of the graph $\gimel = A_\infty$.

Conjecture 3

There exists $\theta_0 \in (0, 2\pi)$ such that for any $\theta \in (0, \theta_0)$, A_∞ has only one arm.

Open Problems

Conjecture 3 is true if one can show the following harder result on the lower bound of growth rate: Define the growth rate of $(A_n)_{n\geq 0}$, denote $\mathrm{gr}((A_n)_{n\geq 0})$ by

$$\mathrm{gr}((A_n)_{n\geq 0}) = \sup\left\{\beta \geq 1/2 : \limsup_{n\to\infty} \frac{\mathrm{diam}(A_n)}{n^\beta} > 0\right\},$$

where diam stands for the diameter of the set in the Euclidean distance.

Conjecture 4

The growth rate is $\mathbb{P}_{\theta_1,\theta_2}$-almost surely a constant and as $\theta \to 0$, it converges to 1.

Related Works:

- E. Procaccia and Y. Zhang, Stationary Harmonic Measure and DLA in the Upper half Plane, *arXiv*: 1711.01011
- E. Procaccia and Y. Zhang, On sets of zero stationary harmonic measure *arXiv*: 1711.01013
- E. Procaccia, R. Rosenthal, and Y. Zhang, Stabilization of DLA in a wedge, *arXiv*: 1804.04236
- E. B. Procaccia, J. Ye, and Y. Zhang, Convergence of two dimensional DLA from a long line segment, near completion
- E. B. Procaccia and Y. Zhang, Two dimensional stationary DLA, in preparation

Thank you!

Questions?

Remarks?